# The Fallacy of Maximizing Risk-Adjusted Returns

Ralph Vince

14 November 2020

*Abstract*

*Routinely, investors in managed funds or those they employ to assess managed funds, do so with certain parameters. Some of these parameters are not performance related (e.g., age of the fund, assets under management, type of fund) whereas there are almost always performance-related measures. Most of these performance measures pertain to perceptions of risk, often in the way of measures of variance with respect to return, worst-case performance depth as well as duration, consistency of returns, etc., collectively often falling under the moniker of "risk-adjusted returns." Herein we demonstrate that, in the limit of continuing time, to maximize risk-adjusted returns is the same exercise as maximizing absolute returns, and hence, an investor in a program that doesn't seek to maximize absolute returns is not getting what he should be getting in terms of the investment fees he pays.*

I. Background

Every trade or trading period which we encounter we will designate as a *proposition*. A proposition is one discrete outcome of a stochastic process comprised of *n* possible outcomes (payoffs) and corresponding probabilities of occurrence.

Thus, we might determine that we are looking at a stream of individual trading outcomes, or a calendar month of outcomes of a specified portfolio, or even something as simple as outcomes of a coin toss; mathematically, we treat this all the same, with only the parameters, the number of possible outcomes, n, and their payoffs and corresponding probabilities being the only variables.

For the sake of simplicity, we consider a coin toss, where a toss of heads yields a 2-unit gain and a toss of tails yields a -1-unit loss, each with corresponding probability of occurrence of 0.5.

We wish to examine what we might expect to make for wagering on this single coin toss with respect to various fractions of our total stake in Figure 1:

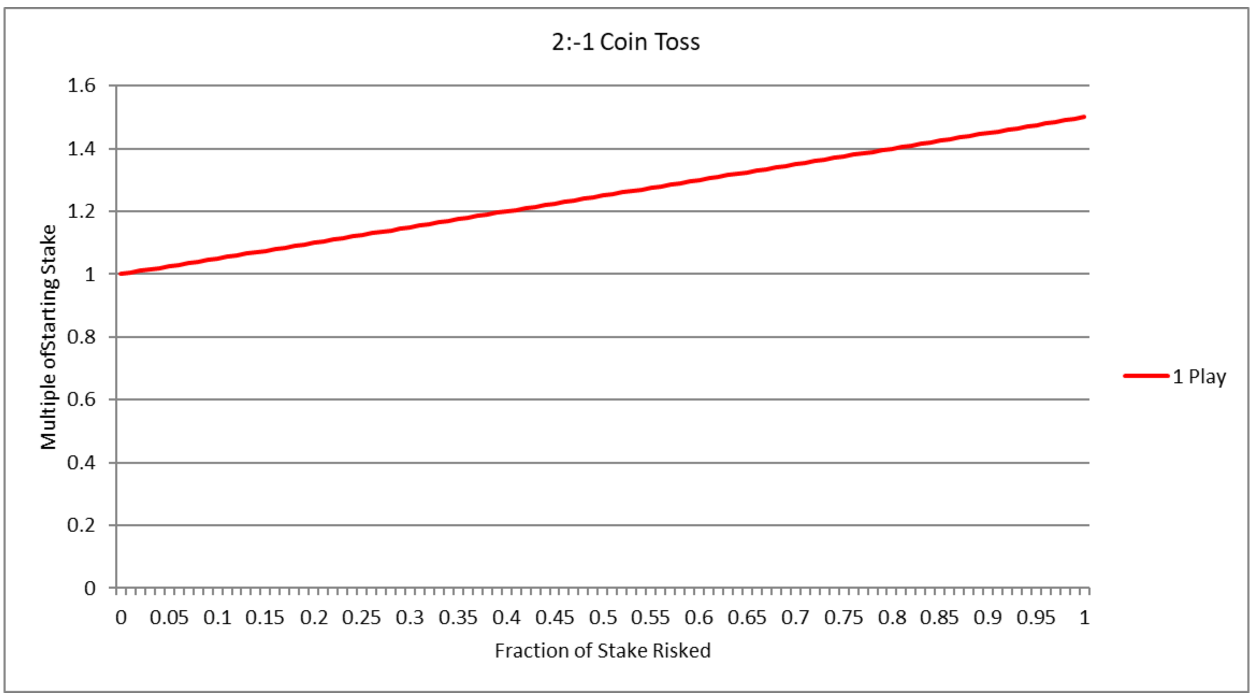


**Figure 1**

And we find if we risk nothing, with 0 along the horizontal axis, we make a multiple of 1 on our stake (we have neither won nor lost anything) and if we risk our entire stake in this 2:-1 coin toss proposition, we expect to make a multiple of 1.5 times our stake.

We now stipulate that all investment funds trade in quantity relative to their size. A 100 billion USD fund will trade in size considerably larger than a fund of a one million USD size.

Thus, as a fund grows (or shrinks) in size in terms of assets under management (AUM), whether by new fund investment or withdrawal, or organically through market performance, so, too, will the trading quantity that the fund works with.

By extension then, we stipulate that returns in managed funds accrue geometrically rather than arithmetically. That is, the return made in this period (which we call TWR, or Terminal Wealth Relative [Vince 1990], expressed as the vertical axis in our graphs) on managed funds is on an amount that was the product of what accrued in previous periods.

Given this, if we look at subsequent propositions beyond the first proposition, we find that the amount we expect to make on our stake, our *TWR*, becomes, therefore, a curved line beginning with the second sequential proposition shown in Figure 2:

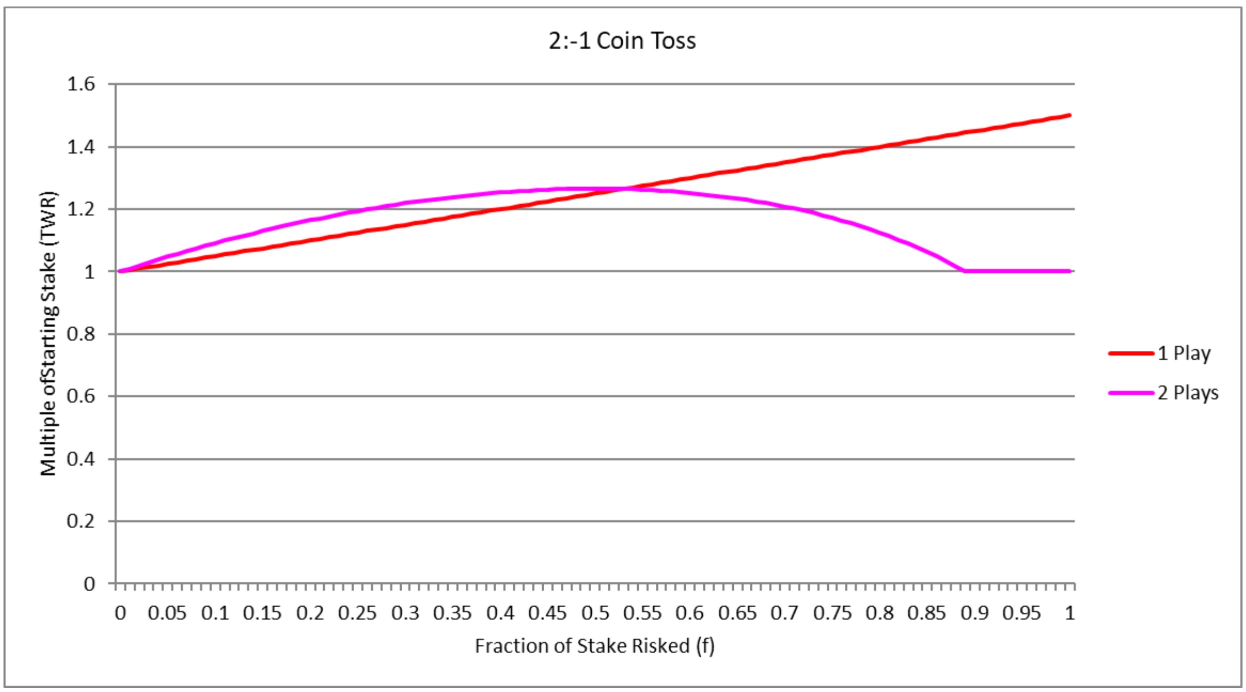


**Figure 2**

For a sequence of 2 plays of this 2:-1 coin toss proposition, we find that the function is now curved with a peak at .5 [Vince 2019] for the parameters $n$=2 and a 2:-1 payoff with probabilities of 0.5 for both outcomes.

We now increase $n$ to 3, for 3 plays of this proposition in Figure 3:

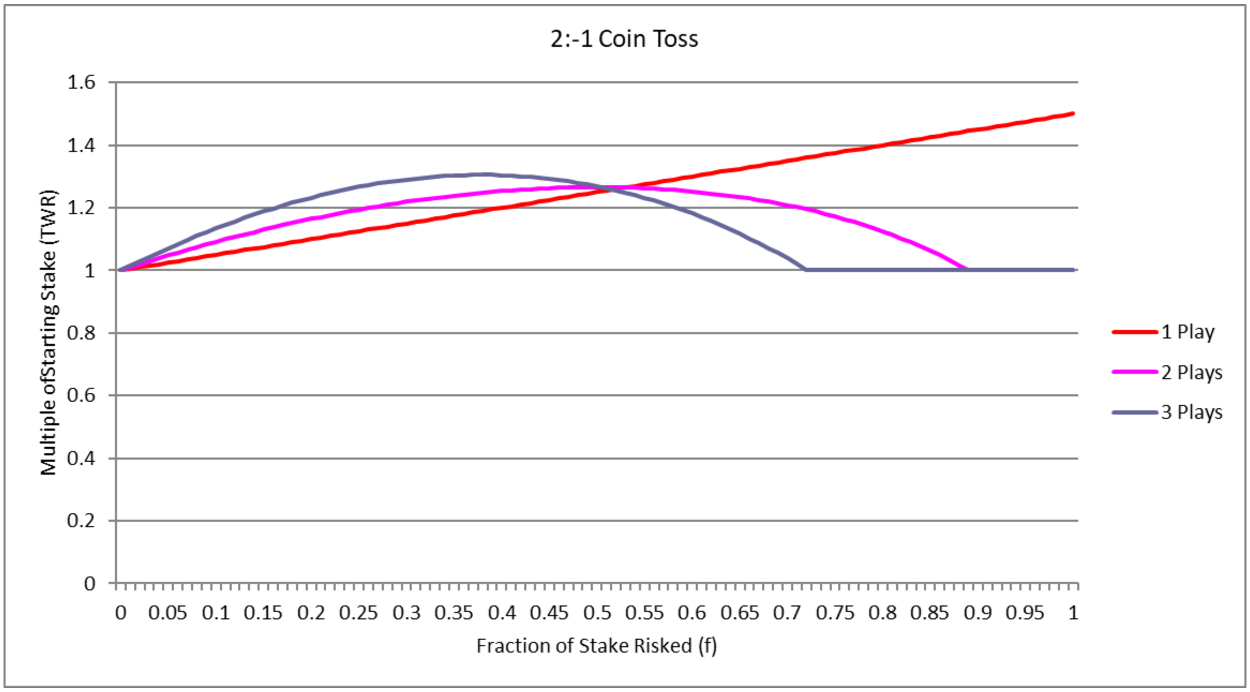


**Figure 3**

We find the curve now at a higher *TWR* at *n=3* and has left-shifted to 0.375.

Let us continue to increase *n* to 8 trials and include 12 trials as well shown in Figure 4:

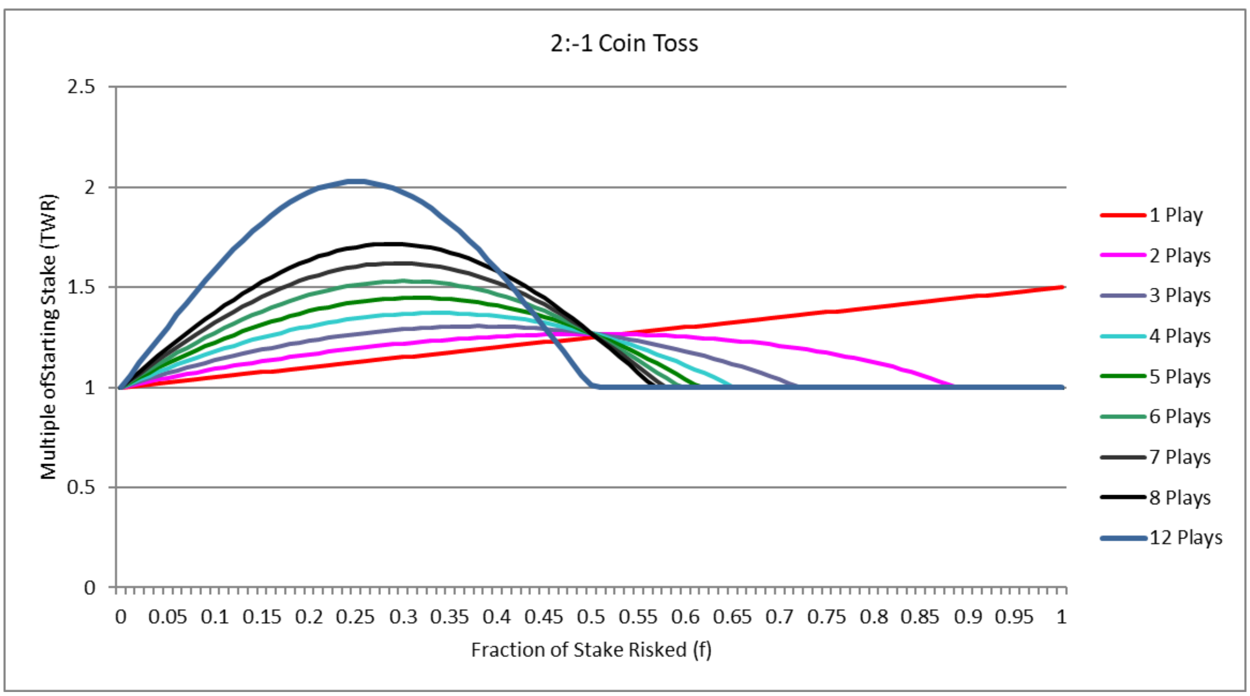


**Figure 4**

Per [Vince 2019], we find the peak not only increasing with each increase in *n*, but we find it settling into its asymptotic fraction, *f*, of 0.25 for the specified parameters of the given proposition. Any further increases in *n* will only approach this asymptotic value for *f* of .25 as *n* increases, yet the peak of the curve will become ever-higher with each increasing value for *n*.

Not only does the peak of the curve get ever-higher, and ever-closer to its asymptotic value for *f*, but just as the shape of the function changed due to reinvestment when we went from one trial, *n=1*, to two subsequent trials, *n=2*, to a curved shape from a straight line one, so, too, does the curve begin to change as *n* gets ever-greater as shown in Figure 5.

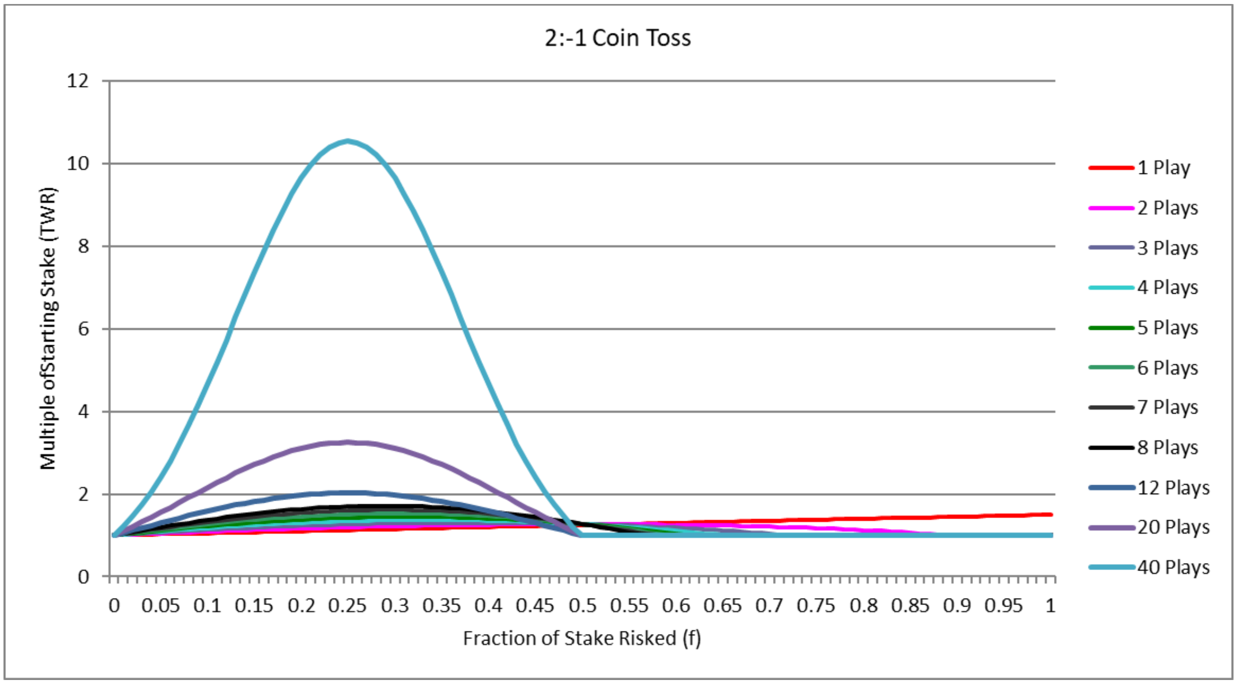


**Figure 5**

Notice at forty plays now, *n=40*, the shape of the curve starts out near zero for f, the vertical axis, as concave up, and as it approaches the peak, shifts to concave down whereupon, after the peak, as we move rightward, *i.e.,* a higher *f*- value, it again flips to concave up and occurs at a value of about 0.13 for *n=40* for these propositions.

This provides two very significant geometric points along the curve with *f*-values less than that corresponding to the peak [de Prado, Vince, Zhu 2019].

The first of these points, the point where we switch from concave-up to concave-down, we refer to as the inflection point. Since reward is presented on the vertical axis, and the percentage we risk along the horizontal axis, then that area left of the peak of the curve represents values for f where gain is increasing at ever-faster rates for each marginal increase in risk. Beyond the inflection point, gain, though still increasing for increasing risk, grows at an ever slower marginal pace for every marginal increase in risk until the peak itself. At the peak itself, growth begins to diminish for any increase in risk shown in Figure 6.

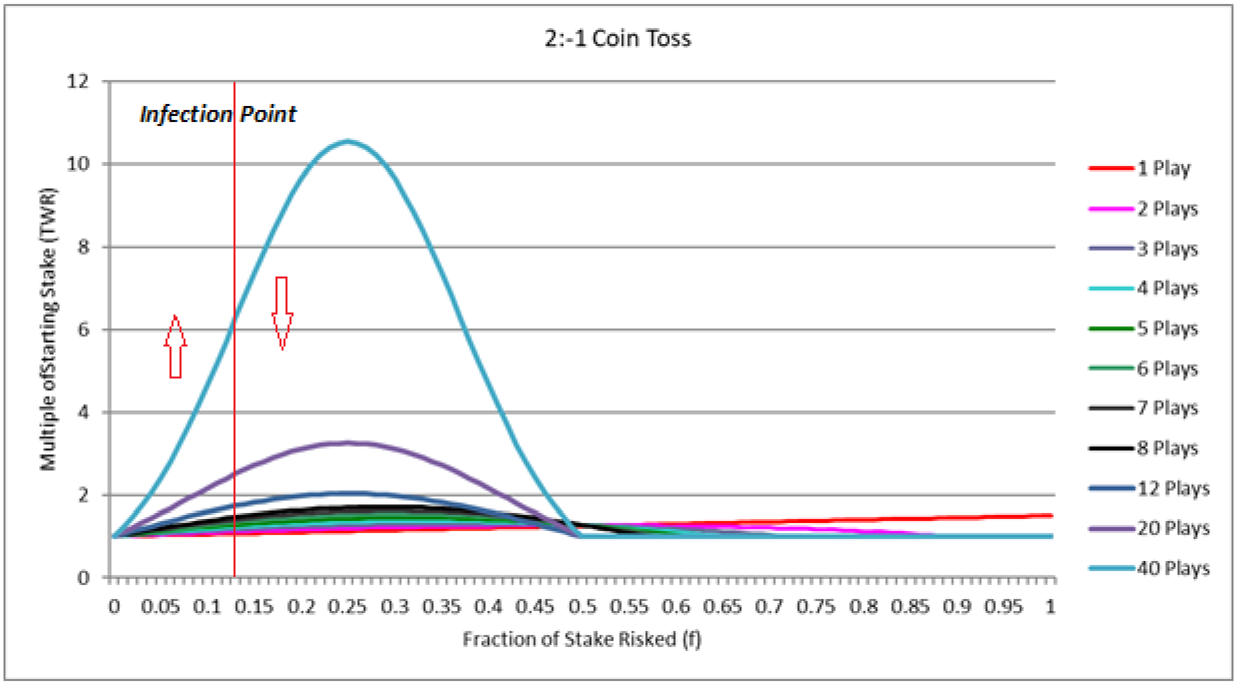


**Figure 6**

Thus, the inflection point represents that point where marginal increase in gain is greatest for ay marginal increase in risk, and is, therefore, one of our two critical points for risk-adjusted returns.

The second geometrically critical point of risk-adjusted returns occurs at a line tangent to the curve from *TWR=1, f=0*. It is the tangent line that possesses the steepest slope of risk-to-reward of any other point on the curve and shown in Figure 7.

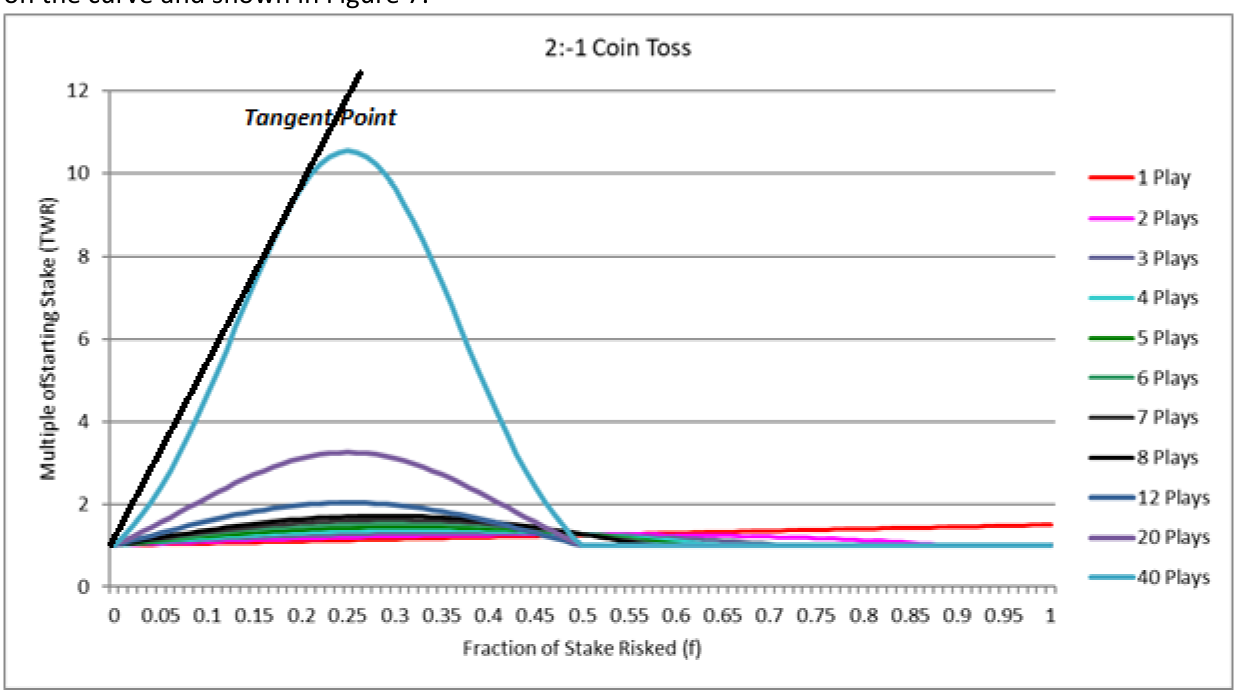


**Figure 7**

For *n=40* with the given parameters, we find the tangent point to be at *f=0.18*.

If we increase n to 100 now, we find dramatic changes shown in Figure 8.

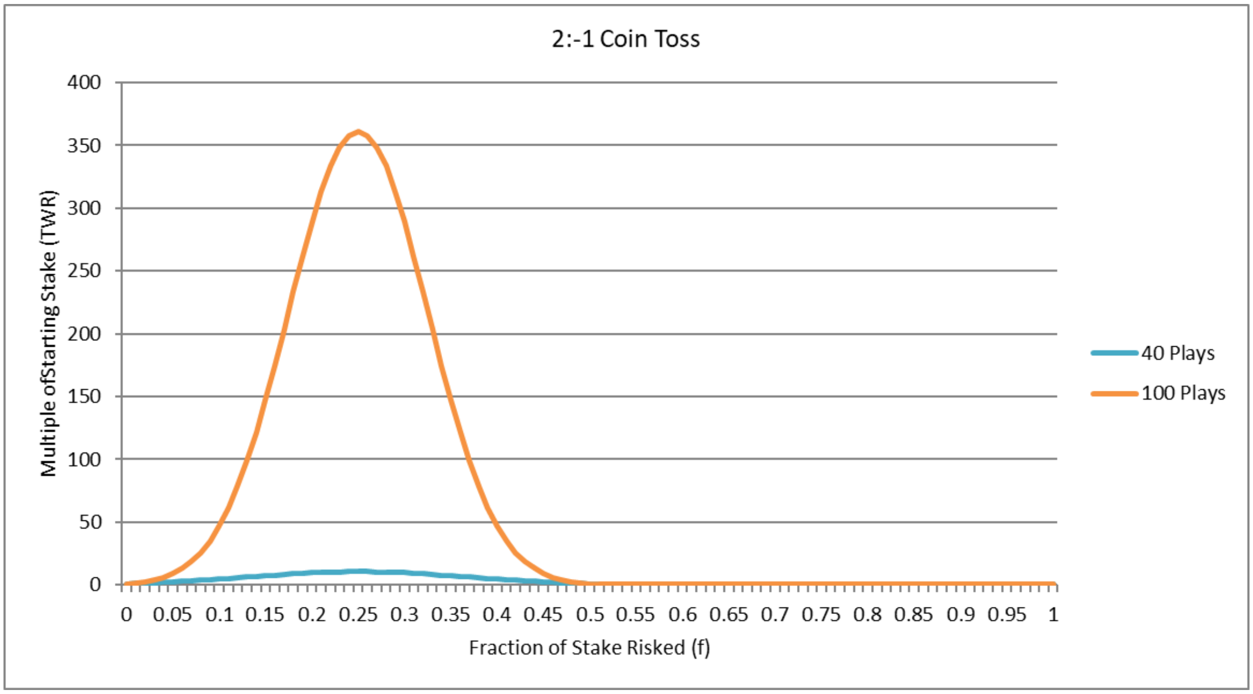


**Figure 8**

First, notice how much higher the peak has become by virtue of increasing *n*. Thus, the very shape of the curve itself has changed. If we examine where the inflection point was at *n=40* at 0.13, it has now moved closer to the peak itself to 0.18 at *n=100* and shown in Figure 9.

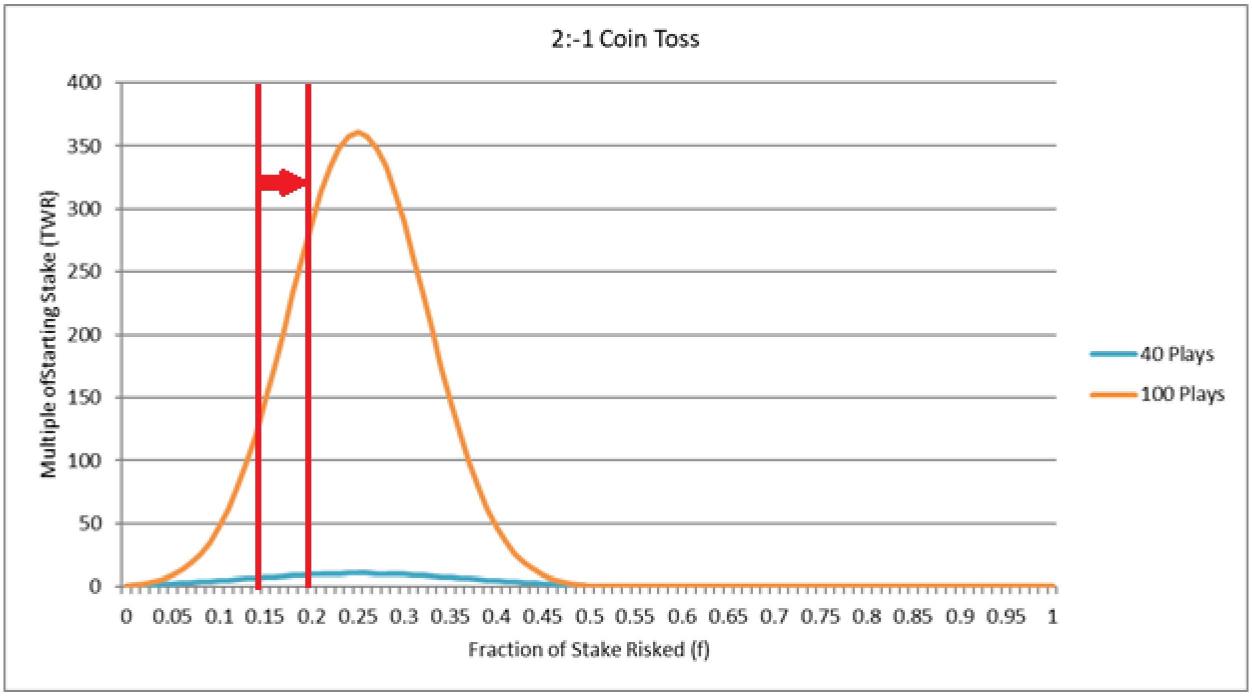


**Figure 9**

Similarly, the tangent point has migrated from 0.18 at *n=40* to .23 at *n=100* as shown in Figure 10.

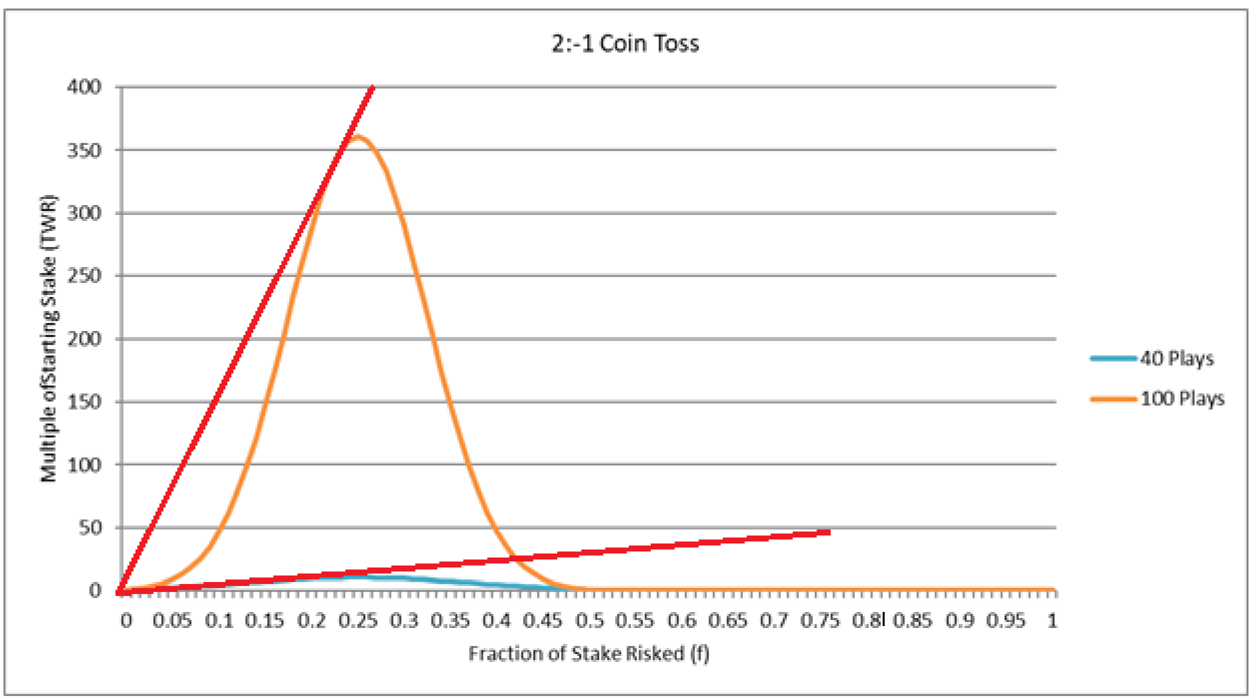


**Figure 10**

As *n* continues to ever-increase, so, too, does the critical point of risk-adjusted return migrate to the value for the asymptotic optimal *f* in the limit as *n* approaches infinity.

Just as demonstrated in [Vince 2019], that the growth optimal fraction, *i.e.*, the value for *f* corresponding to the peak of the curve, asymptotically migrates to a specified point, the asymptotic value for *f* (which, in *special cases*[1], equals the same value as that which satisfies the Kelly Criterion for the given parameters), so, too, do these two points of critical geometry specific to maximizing risk-adjusted returns migrate to that same specified point, shown in the table below and in Figure 11.

| *n* | *f* | *inflection* | *tangent* |
|---|---|---|---|
| 1 | 1.00 | - | - |
| 2 | 0.50 | - | - |
| 3 | 0.38 | - | - |
| 4 | 0.34 | - | - |
| 5 | 0.31 | - | - |
| 6 | 0.30 | - | - |
| 7 | 0.29 | - | - |
| 8 | 0.28 | - | - |
| 12 | 0.27 | 0.03 | 0.04 |
| 20 | 0.26 | 0.08 | 0.11 |
| 40 | 0.25 | 0.13 | 0.18 |
| 100 | 0.25 | 0.18 | 0.23 |
| 1000 | 0.25 | 0.23 | 0.25 |

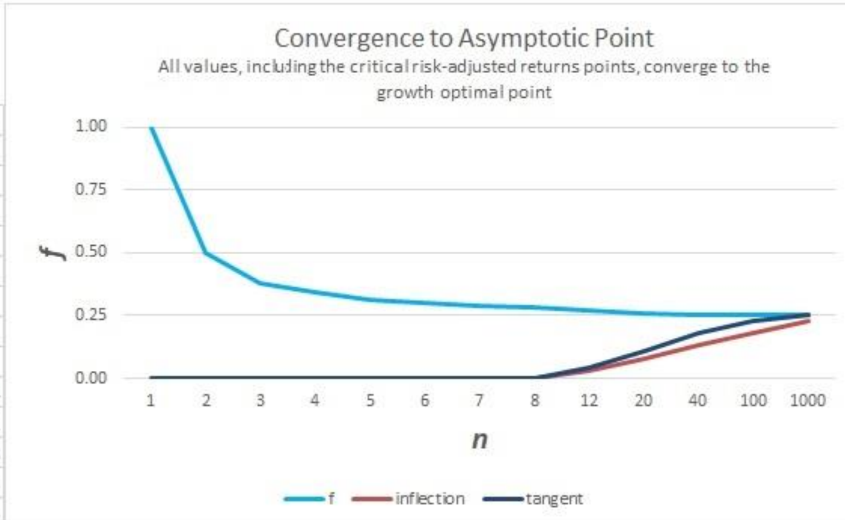


**Figure 11**

As mentioned earlier, although we have used a proposition of a coin toss for simplicity, all propositions involving risk comport to this same analysis, the only difference being, given the different parameters of different risk-based propositions are the possible outcomes and respective probabilities of occurrence which yield a peak to the curve which ultimately settles in to an asymptotic value less than 1.0, and differing values for *n* where an inflection point appears in the curve, and hence a tangent point as well.

Next, we examine measures of return with respect to drawdown or other ersatz measures of equity retracement depth and/or duration. In [Vince 2007], it has been demonstrated that a drawdown of *any* given magnitude has a probability of occurring that approaches 1 as the number of periods or trials increases. The speed at which the probability closes in on this asymptotic value of 1.0, certainty of occurring, is a function of the degree of drawdown being examined, the character of the market approach, the degree of leverage used (*i.e.*, "*f*"), and the random nature of loss sizes and bunching. Regardless, as time accrues, the probability of a drawdown of any magnitude approaches 1. To ascribe the value for these metrics' importance is to make the mistake that the character of the market approach behind the value is the sole determinant when it is usually the least deterministic.

[1] Such cases include where the amount risked is the same as what can be lost (thus, excludes the game of Blackjack) but additionally where the probability-weighted mean outcome is positive. If either of these conditions are not met, the Kelly Criterion solution will either not exist or will not equal the **asymptotic** growth optimal fraction to risk. If both cases are met, the Kelly Criterion solution will only return the asymptotic growth optimal limit that is approached.

Such metrics reflect more the level of leverage used (usually an *ad-hoc* selection) and the time it has been observed over.

Often, funds are assessed on such *ad-hoc* and quite relatively baseless metrics which measure return vis-à-vis variance in those returns. These types of measures, including the Sharpe, Sortino, and Treynor ratios, and other such measures, typically measure dilution of what would be the growth-optimal trading level. Worse-still, these metrics are often easily "gamed."

In [Vince 1992], we see in terms the arithmetic mean (*A*), geometric mean (*G*) and standard deviation (*SD*) in holding period returns for various degrees of leverage (*i.e.*, values for *f*, bound between 0 and 1), and comport very closely [Vince 2020[ to the Pythagorean Theorem shown in Figure 12.

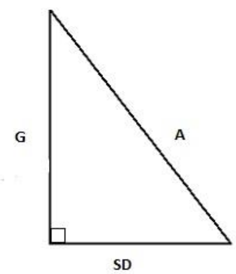


**Figure 12**

*Pythagorean relationship of* $A^2=G^2+SD^2$.

When we consider that at *f=0*, both *G* and *A = 1* and *SD = 0*. As we progress to higher levels of leverage, higher values for *f*, these values change such that *A* (the hypotenuse), along with *SD*, continue to increase while *G* increases only to the peak, whereupon it begins to decrease shown in Figure 13.


**Figure 13**

Thus, we can say there is an isomorphism that exists, a mapping between *f* and the ratio of *SD* to *A*, the cosine, shown in in Figure 14.


**Figure 14**

. Since measures of risk to variance are what we are considering, this would thus be the reciprocal of the cosine of $SD^2$ to *A*, or, specifically such measures are therefore the secant ($A / SD^2$) which is the isomorphism to *f* as represented by such measures (results given in radians or degrees) and thus exhibits how measures such as the Sharpe ratio are isomorphisms to *f* for the horizontal axis.

In other words, ***measures of return to variance in returns are simply a different mapping than f for the same axis!*** *They are non-dimensional measures of leverage which map to the dimensional measure of* *leverage, f, being isomorphisms to f, the horizontal axis. To compare programs based on these measures is to compare them based on the (unknown) f-values used. It is indicative of the leverage used, not the virtue of the program itself. In effect, you are comparing two different programs and two unknown levels of leverage, with the latter holding most of the sway of such metrics*

Conclusion

We see that points comparing returns to drawdowns essentially compare systems of differing, unknown values of leverage ("*f*") (and time under scrutiny) rather than an apples-to-apples comparison of risk-adjusted returns which the assessor seeks. We further find that measures that compare returns to variance essentially compare different systems at different (unknown) levels of leverage. Again, this is not an apples-to-apples comparison of the merits of an approach's equity curve but of the merits of the different and unknown levels of leverage. What those who are using these metrics for purposes of comparing the relative merits of different market approaches are comparing is more the differences in leverage employed than the characteristics of the program aside from the leverage employed.

We have seen that there is no benefit to leverage employed beyond that represented at the peak of the curve. We find, aside from the peak of the curve where growth is maximized, there exists but two non-arbitrary points of risk-adjusted return maximization. These two points are the inflection and the tangent points. Any other points on the curve are insignificant in terms of risk-adjusted return maximization and, therefore, *ad-hoc*.

These two points of risk-adjusted return maximization do not exist for small values of $n$, *i.e.*, a small number of trials. In fact, at one trial, the curve is not a curve at all but an upward sloping straight line. Only when reinvestment is considered, that is, when a subsequent trial is encountered, does the curve bend. Further, as $n$ increases, at some value for $n$ (as a function of the parameters of the proposition's outcomes and corresponding probabilities) an inflection point emerges, and hence a tangent point (greater than at $f=0$) also emerges with it.

Once this transpires, risk-adjusted returns can be maximized for a known number of trials, a pre-determined $n$, for the proposition's parameters. However, painfully, these two points of risk-adjusted return maximization, the inflection point and the tangent point, migrate to the growth-optimal peak itself as $n$ grows ever greater. Therefore, for an unknown $n$ where the implementor intends to continue trials *ad infinitum*, the points that maximize risk adjusted returns asymptotically approach the same point, the same level of leverage, that maximizes compound growth.

Hence it can be said that an investor who has put his money into a program being traded at levels of leverage that are less than growth optimal is not getting what he is paying for in terms of fees. He should be paying for a smaller amount of investment, and charged fees on this smaller investment, in the same program traded at the growth-optimal level of leverage. Absent this, he is, in effect, paying the manager to "invest" a large portion of his investment in a risk-free type asset and being charged full management fees on that, something the investor can do himself without being charged such fees.

## References

M. Lopez de Prado, R. Vince, and Q. Zhu. Optimal Risk Budgeting Under a Finite Investment Horizon, Risks 7(3), 2019, 86.

R. Vince. Portfolio Management Formulas. John Wiley and Sons, New York, 1990.

R. Vince. The Mathematics of Money Management: Risk Analysis Techniques for Traders. John Wiley and Sons, New York, 1992.

R. Vince. The Handbook of Portfolio Mathematics: Formulas for Optimal Allocation and Leverage. John Wiley and Sons, Hoboken, NJ, 2007.

R. Vince and Q. J. Zhu. Optimal Betting Sizes for the Game of Blackjack. Journal of Investment Strategies 4(4), 2015, 53-75.

R. Vince. Expectation and Optimal *f*. Far East Journal of Theoretical Statistics. 56(1), 2019, 69 - 91.

R. Vince. Diminution of Malevolent Geometric Growth Through Increased Variance Journal of Economics, Business and Market Research, Volume 1(1), Pages 35 - 53 (June 2020)