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Many-to-one matchings without substitutability

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ABSTRACT

We study the existence of stable matchings in the many-to-one college admission problem when there are no restrictions on college preferences. We show that the existence of a stable allocation is strongly tied to students having "*harmonious preferences*" over their sets of acceptable college choices. In other words, without any assumption on college preferences, a stable matching exists for any college admission problem if and only if there is no subset of students with misaligned college rankings.

1. Introduction

When college preferences satisfy substitutability, the existence of a stable matching is guaranteed.[1] In actuality, however, we observe stable allocations where colleges have non-substitutable preferences over students. College admissions are subject to concerns such as affirmative action and diversity in the form of quota requirements regarding gender and race.[2] We observe similar preference structure in the labor market as well. Office of Federal Contract Compliance Programs (OFCCP) enforces "*the contractual promise of affirmative action and equal employment opportunity required of those who do business with the Federal government*". As of 2014, about 20% of total US civilian workforce is subject to the OFCCP.[3]

Different from the earlier works, we focus on the necessary and sufficient conditions on the preference profile of the students which guarantee the existence of a stable matching when there are no restrictions on college preferences. We prove that a stable matching exists if and only if there is no subset of students with misaligned college rankings.[4]

Empirical evidence suggests that students have commonly ranked preferences over the schools they want to apply. Publicly available sources such as US News College Rankings, and the historical reputation of schools lead to a nearly uniform formation of students' preferences. We see similar patterns in the labor market as well. Publicly available "Best companies to work for" lists lead employees to have common information regarding employers' characteristics.[5]

Parchment, an online platform for students and colleges, launched a "colleges ranking system" based on revealed preferences of students.[6] We collected the Parchment revealed preference data,[7] and compared it to the US News rankings. We show that whenever admitted to two colleges, 86% of the time students picked the school ranked higher by US News when having to choose between two schools located in the same state, and one is ranked inside the top 10 while the other outside the top 10 according to the US News rankings.[8]

Griffith and Rask (2007) estimate that students' choices are responsive to changes in schools' US News ranks, and this effect is stronger among higher ranked schools. Similarly, Bowman and Bastedo (2009) find that moving onto the front page of the US News rankings increase the following year's application/admission indicators substantially. Luca and Smith (2013) find that a one-rank

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1 See Kelso and Crawford, 1982; Roth and Sotomayor, 1990; Hatfield and Milgrom, 2005; Echenique and Oviedo, 2006; Pycia, 2012.

2 See Long (2004) and Chan and Eyster (2003).

3 http://www.dol.gov/ofccp/PayTransparencyFactSheet.html.

4 Hałaburda (2010) analyzes factors for unraveling in many-to-one matchings between firms and workers, assuming identical worker preferences over firms. Ekmekci and Yenmez (2014) analyzes schools' incentives to join clearinghouses when students have identical preferences over schools.

5 http://www.businessinsider.com/50-best-companies-to-work-for-in-america-methodology-2015-4.

6 http://www.parchment.com/c/college/college-rankings.php.

7 http://www.nytimes.com/interactive/2014/09/04/upshot/college-picks.html?_r=0&abt=0002&abg=1.

8 See Data Appendix.

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improvement in US News leads to a 1% increase in the number of applications.

The closest work to ours, Pycia (2012) shows that there is a stable coalition structure, for all preference profiles if and only if agents' preferences are pairwise aligned, i.e. if any two agents rank proper coalitions that contain both of them in exactly the same way. The solution to the complementary problem behind Pycia's and our findings is very similar in nature: a commonality across preferences. Our preference domain, however, allows students to be indifferent about their potential future classmates whereas still letting colleges have any preferences over the sets of applicants.

Echenique and Yenmez (2007) analyze many-to-one matchings where students have preferences over other students. The authors introduce a new method that finds all core matchings. Similarly, Dutta and Masso (1997) allow some peer effects (lexicographic preferences) but maintained substitutability on the colleges' preferences. Kojima et al. (2013) analyze why stability exists empirically under circumstances where the theory suggests it should not. They identify conditions under which a stable matching exists with high probability in large markets. We analyze the very same problem of stability by using the similarity of student preferences.

2. Model

We study the college admission problem (Gale and Shapley, 1962; Roth and Sotomayor, 1990), represented by $(C, S, q, \succ_S, \succ_C)$:

1. set of colleges C;
2. set of students S;
3. capacity vector $q = (q_c)_{c \in C}$, where q_c is the capacity of $c \in C$;
4. preference profile for students $\succ_S = (\succ_s)_{s \in S}$, where $\succ_s$ is student s's strict preference over colleges and being unassigned option, $\emptyset$;
5. preference profile for colleges $\succ_C = (\succ_c)_{c \in C}$, where $\succ_c$ is college c's strict preference over the subsets of S.

Let $\succsim_i$ be the at-least-as-good-as relation with $\succ_i$ for all $i \in C \cup S$. College $c \in C$ is unacceptable for student s if $\emptyset \succ_s c$. We fix q and represent the problem with $(C, S, \succ_S, \succ_C)$.

A **matching** is a correspondence $\mu : C \cup S \twoheadrightarrow C \cup S \cup \emptyset$ such that:

- $\mu(c) \subseteq S$ where $|\mu(c)| \leq q_c$ for all $c \in C$,
- $\mu(s) \subseteq C$ where $|\mu(s)| \leq 1$ for all $s \in S$,
- $s \in \mu(c)$ if and only if $\mu(s) = c$ for all $c \in C$ and $s \in S$.

Let $\mathcal{M}$ be the set of matchings.

A matching μ is **blocked by student** $s \in S$ if $\emptyset \succ_s \mu(s)$. A matching μ is **blocked by college** $c \in C$ if there exists $\tilde{S} \subset \mu(c)$ such that $\tilde{S} \succ_c \mu(c)$. A matching is **individually rational** if it is not blocked by any individual agent. A matching μ is **blocked by the college-student pair** (c, s) if $c \succ_s \mu(s)$ and there exists $\tilde{S} \subseteq \mu(c) \cup \{s\}$ such that $s \in \tilde{S}$, $\tilde{S} \succ_c \mu(c)$ and $|\tilde{S}| \leq q_c$. A matching μ is (**pairwise**) **stable** if it is not blocked by any individual agent or any college-student pair.

We say a matching μ is **blocked by a coalition** K if there exist another matching $\nu \in \mathcal{M}$ and a coalition $K \subset C \cup S$, such that $\{c\} = C \cap K$ and for all $s \in S \cap K$: (1) $\nu(s) = c$ and $\nu(s) \succ_s \mu(s)$, (2) $K \subseteq \nu(c) \subseteq K \cup \mu(c)$ and $\nu(c) \succ_c \mu(c)$. A matching μ is **group stable** if it is not blocked by any coalition.

A **mechanism** ψ specifies a matching for each problem. Outcome of ψ in $(C, S, \succ_S, \succ_C)$ is $\psi(C, S, \succ_S, \succ_C)$ and the assignment of $i \in S \cup C$ is $\psi_i(C, S, \succ_S, \succ_C)$. A mechanism is (group) stable if it selects a (group) stable matching in any problem.

Next, we define harmonious preferences.

Definition 1. Preferences of students $\{s_1, s_2, \ldots, s_k\} \subseteq S$ are **inharmonious** if there exists $\{c_1, c_2, \ldots, c_k\} \subseteq C$ such that $c_m \succ_{s_m} c_{m-1} \succ_{s_m} \emptyset$ for all $m \in \{1, 2, \ldots, k\}$ where $c_0 = c_k$.

We say $\succ_S$ is **harmonious** if there don't exist s and $\bar{s}$ with inharmonious preferences. Students may have different yet still harmonious preferences.

We present the following example motivating the need for a general set of college preferences.

Example 1. Let $S = \{s_1, s_2, s_3, s_4, s_5\}$, $C = \{c_1\}$, $q_{c_1} = 3$ and $\{s_1\} \succ_{c_1} \{s_2\} \succ_{c_1} \{s_5\} \succ_{c_1} \{s_3\} \succ_{c_1} \{s_4\}$. c_1 applies the following admission policy: whenever available, each cohort has at least one female and one Hispanic student, and for the remaining seats its preferences are responsive over the available candidates. Consider the following gender/race composition: s_1 and s_2 are non-Hispanic males, s_3 is Hispanic male, s_4 is non-Hispanic female and s_5 is a Hispanic female. Then, c_1 selects $\{s_1, s_3, s_4\}$ and $\{s_1, s_2, s_5\}$, from $\{s_1, s_2, s_3, s_4\}$ and $\{s_1, s_2, s_3, s_4, s_5\}$, respectively. Hence, c_1's preferences violate the substitutability assumption.

3. Results

We prove our main result in two steps. First, Proposition 1 shows that harmonious student preferences constitute a necessary condition. Second, using Lemma 1 and Proposition 2, we prove that harmonious student preferences guarantee the existence of a stable matching.

Proposition 1. *Given $C, S, \succ_S$, if there exists a subset of students $\bar{S} \subseteq S$ with inharmonious preferences, then there exists $\succ_C$ such that a stable matching does not exist under $(C, S, \succ_S, \succ_C)$.*

Proof. Let $\bar{S} = \{s_1, s_2, \ldots, s_k\}$, $\bar{C} = \{c_1, c_2, \ldots, c_k\}$, and $c_m \succ_{s_m} c_{m-1} \succ_{s_m} \emptyset$ for all $m \in \{1, 2, \ldots, k\}$ where $c_0 = c_k$. Let $q_{c_1} = 2$ and $q_a = 1$ for all $a \in C \setminus \{c_1\}$. College preferences are: $\{s_1, s_2\} \succ_{c_1} \{s_2\} \succ_{c_1} \emptyset \succ_{c_1} \tilde{S}$ for all $\tilde{S} \in 2^S \setminus \{\{s_1, s_2\}, \{s_2\}, \{\emptyset\}\}$, $\{s_{m+1}\} \succ_{c_m} \{s_m\} \succ_{c_m} \emptyset \succ_{c_m} \tilde{S}$ for all $m \in \{2, 3, \ldots, k-1\}$ and $\tilde{S} \in 2^S \setminus \{\{s_m\}, \{s_{m+1}\}, \{\emptyset\}\}$; $\{s_1\} \succ_{c_k} \{s_k\} \succ_{c_k} \emptyset \succ_{c_k} \tilde{S}$ for all $\tilde{S} \in 2^S \setminus \{\{s_1\}, \{s_k\}, \{\emptyset\}\}$, $\emptyset \succ_{c'} S'$ for all $S' \in 2^S$ with $S' \cap \bar{S} \neq \emptyset$ and $c' \in C \setminus \bar{C}$. On contrary, suppose there exists a stable matching denoted by μ. Under μ, students in $\bar{S}$ are assigned to either their top or second choice, and students in $S \setminus \bar{S}$ cannot be assigned to colleges in $\bar{C}$. Otherwise, μ cannot be stable. Suppose $\mu(s_2) = c_1$. Then, $\mu(s_1) = c_1$. Otherwise, (c_1, s_1) would block μ. This implies that $\mu(s_m) = c_m$ for all $m \in \{3, 4, \ldots, k\}$. Then, $\mu(c_2) = \emptyset$ and (c_2, s_2) would block μ. Now suppose $\mu(s_2) \neq c_1$. Then, $\mu(s_1) \neq c_1$. Otherwise, c_1 would block μ. Hence, $\mu(s_1) = c_k$ and $\mu(s_m) = c_{m-1}$ for all $m \in \{3, 4, \ldots, k\}$. Then, $\mu(c_1) = \emptyset$, $\mu(s_2) = \emptyset$ and (c_1, s_2) would block μ. This is a contradiction. ∎

Next, we show that the student proposing deferred acceptance (DA) mechanism (Gale and Shapley, 1962) doesn't pick a stable matching under our setting.

Example 2. Let $C = \{c_1, c_2\}$, $S = \{s_1, s_2, s_3\}$, $q_{c_1} = 2$, $q_{c_2} = 1$, $\{s_1, s_2\} \succ_{c_1} \{s_2\} \succ_{c_1} \emptyset \succ_{c_1} \{s_1\}$, $\{s_3\} \succ_{c_2} \{s_2\} \succ_{c_2} \emptyset$, $c_1 \succ_{s_1} \emptyset \succ_{s_1} c_2$, $c_2 \succ_{s_2} c_1 \succ_{s_2} \emptyset$ and $c_2 \succ_{s_3} \emptyset \succ_{s_3} c_1$. $\succ_S$ is harmonious. DA selects $\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & c_1 & c_2 \end{pmatrix}$ which is blocked by (c_1, s_1). Moreover, there exists a stable matching: $\nu = \begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_1 & c_2 \end{pmatrix}$.

Next lemma helps us to show that the mechanism to be used for the sufficiency condition is well-defined.

Table A.1
US news rankings v. revealed student preferences.

	All	Same state	Schools ranked <25	Schools ranked <25 (Same state)	Top 25 vs. Outside 25	Top 25 vs. Outside 25 (same state)	Top 10 vs. Outside Top 10	Top 10 vs. Outside Top 10 (Same state)	Top 5 vs. Outside Top 5	Top 5 vs. Outside Top 5 (Same state)
Sample size	2987	596	2474	378	511	218	1182	141	1055	152
In favor of US news	1969	397	1717	254	342	143	985	121	849	125
US news consistency	68.4%	66.6%	69.4%	67.2%	66.9%	65.6%	83.3%	85.8%	80.5%	82.2%

Lemma 1. *If $\succ_S$ is harmonious, then there exists $c \in C$ which is ranked at the top of the preferences of each student considering it acceptable.*

Proof. On the contrary, suppose there does not exist such a college. Consider an arbitrary student $s_1 \in S$ and denote her top choice with c_1. Let S_1 be the set of students considering c_1 as the best school. Note that $s_1 \in S_1$ and $|S_1| > 0$. There exists (c_2, s_2) such that $c_1 \neq c_2$, $c_2 \succ_{s_2} c$ for all $c \in C \setminus \{c_2\}$ and $c_1 \succ_{s_2} \emptyset$. Otherwise, c_1 is the best college for all students considering it acceptable. By definition, $s_2 \in S \setminus S_1$. Let S_2 be the set of students considering c_2 as the best college. Note that $S_2 \subseteq S \setminus S_1$ and $|S_2| > 0$. There exists (c_3, s_3) such that $c_3 \neq c_2$, $c_3 \succ_{s_3} c$ for all $c \in C \setminus \{c_3\}$ and $c_2 \succ_{s_3} \emptyset$. Otherwise, c_2 is the best college for all students considering it acceptable. Since $\succ_s$ is harmonious, $c_3 \neq c_1$. Hence, $s_3 \in S \setminus (S_1 \cup S_2)$. By the same argument, if none of the colleges is ranked as a top choice by all students considering it acceptable, then there exists (c_{k+1}, s_{k+1}) such that $c_{k+1} \notin \{c_1, c_2, \ldots, c_k\}$, $s_{k+1} \in S \setminus (\cup_{\ell=1}^{k} S_\ell)$, $c_{k+1} \succ_{s_{k+1}} c$ for all $c \in C \setminus \{c_{k+1}\}$ and $c_k \succ_{s_{k+1}} \emptyset$ where S_ℓ is the set of students ranking c_ℓ at the top of their preference list. Since the number of students is finite, $\cup_{\ell=1}^{m} S_\ell = S$ for some $m \in \mathbb{N}$. This is a contradiction. ■

In Lemma 1, we show that whenever the student preference profile is harmonious then there exists at least one college that is ranked at the top of the preference lists of all students who consider that college acceptable. Let $\Phi_c(C, S, \succ_S, \succ_C)$ be the set of students considering c acceptable and preferring it over all the other colleges, i.e., $\Phi_c(C, S, \succ_S, \succ_C) = \{s \in S | c \succ_s x \text{ for all } x \in (C \cup \{\emptyset\}) \setminus \{c\}\}$. Let $T(C, S, \succ_S, \succ_C)$ be the set of colleges that are ranked at top by students who consider them acceptable, i.e., $T(C, S, \succ_S, \succ_C) = \{c \in C | \emptyset \succ_s c \text{ for all } s \notin \Phi_c(C, S, \succ_S, \succ_C)\}$. Next, we define our mechanism.

Top College Picking (TCP) Mechanism

Step 1: Students in S considering all colleges in C unacceptable are assigned to $\emptyset$. Each $c \in T(C, S, \succ_S, \succ_C)$ is assigned to the highest ranked subset of students in $\Phi_c(C, S, \succ_S, \succ_C)$. Assigned students and colleges are removed. Denote the remaining colleges and students with C_1 and S_1, respectively. In general:

Step k > 1: Students in S_{k-1} considering all colleges in C_{k-1} unacceptable are assigned to $\emptyset$. Each $c \in T(C_{k-1}, S_{k-1}, \succ_{S_{k-1}}, \succ_{C_{k-1}})$ is assigned to the highest ranked subset of students in $\Phi_c(C_{k-1}, S_{k-1}, \succ_{S_{k-1}}, \succ_{C_{k-1}})$. Assigned students and colleges are removed. Denote the remaining colleges and students with C_k and S_k, respectively.

TCP terminates whenever the set of remaining students is empty. TCP can be considered as a version of the serial dictatorship mechanism. However, the order of the colleges choosing in each step depends on the preferences of the students.

By Lemma 1, when students' preference profile is harmonious the TCP is well-defined. Next, we present the sufficiency condition in Proposition 2.

Proposition 2. *If $\succ_S$ is harmonious, then TCP selects the unique group stable outcome in problem $(C, S, \succ_S, \succ_C)$.*

Proof. Let μ be outcome of TCP in $(C, S, \succ_S, \succ_C)$. Let S^k and C^k be the set of students and colleges removed in Step k of TCP.

Group stability: Each $s \in S$ can be assigned to $c \in C$ if $c \succ_s \emptyset$. Moreover, there are no $c \in C$ and $\tilde{S} \subset \mu(c)$ such that $\tilde{S} \succ_c \mu(c)$. Otherwise, c would have only selected the students in $\tilde{S}$ in the step it was removed. Hence, μ isn't individually blocked. Suppose there exist $c \in C$ and $\tilde{S} \subseteq S$ blocking μ, and $c \in C^k$. That is, c is removed in step k. Since μ is individually rational for each $s \in \tilde{S}$, $c \succ_s \mu(s) \succsim_s \emptyset$. Hence, $c \succ_s \emptyset$ for each $s \in \tilde{S}$. Since in every step, removed students are assigned to their top choices among the remaining colleges, $s \in S^{k' > k}$ for all $s \in \tilde{S}$. Since $\succ_S$ is harmonious, in step k in which c selects its most preferred subset, students considering c acceptable must be in $\Phi_c(C_{k-1}, S_{k-1}, \succ_{S_{k-1}}, \succ_{C_{k-1}})$, i.e., $\tilde{S} \subseteq \Phi_c(C_{k-1}, S_{k-1}, \succ_{S_{k-1}}, \succ_{C_{k-1}})$. By definition of TCP, c prefers $\mu(c)$ to any subset of $\mu(c) \cup \tilde{S}$. This is a contradiction.

Uniqueness: Consider a matching $\mu' \in \mathcal{M}$ such that $\mu' \neq \mu$. By induction, we prove that μ' cannot be group stable. Consider the colleges assigned in the first step. Suppose $c \in C^1$ and $\mu(c) \neq \mu'(c)$. All students considering c acceptable rank it at the top. Hence, all students in $\mu'(c)$ rank c first. Otherwise, individual rationality would be violated. Then, $\mu'(c) \subseteq \Phi_c(C, S, \succ_S, \succ_C)$. By definition of TCP, $\mu(c)$ is the most preferred subset of students for c in $\Phi_c(C, S, \succ_S, \succ_C)$, i.e., $\mu(c) \succ_c \mu'(c)$. Hence, μ' will be blocked by students in $\mu(c) \setminus \mu'(c)$ and c. Hence $\mu(c) = \mu'(c)$ for all $c \in C^1$. Suppose $\mu(c) = \mu'(c)$ for all $c \in \cup_{t=1}^{k} C^t$. We show that $\mu(c) = \mu'(c)$ for all $c \in C^{k+1}$. Among available students in Step $k+1$, only the ones who prefer c the most compared to the remaining colleges consider c acceptable. Hence, all students in $\mu'(c)$ prefer c the most compared to the remaining colleges in Step $k+1$. Otherwise, individual rationality is violated. By definition of TCP, $\mu(c)$ is the most preferred subset of students for c among the ones preferring c most in step $k+1$, $\mu(c) \succ_c \mu'(c)$. Hence, μ' will be blocked by students $\mu(c) \setminus \mu'(c)$ and c. ■

Theorem 1. *When there are no restrictions on college preferences, a stable matching exists for any problem if and only if the student preference profile is harmonious.*

Proof. Follows from Propositions 1 and 2. ■

Appendix. Data appendix

Our sample contains all pairwise decisions where at least one of the admitted colleges were ranked inside top 25 by US News. Our calculations are consistent with US News whenever a student is accepted to both schools. Table A.1 summarizes our findings.

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